Lecture 3

Summary of the last two lectures

(1) one can get order of composite (bilinear) combinations of the primary order parameter

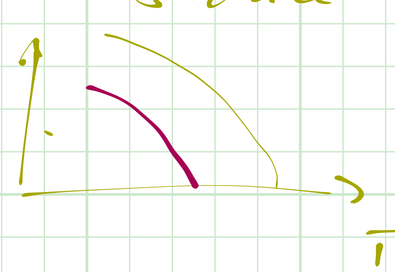
$$\Gamma \otimes \Gamma = \Gamma_1 \oplus \underbrace{\Gamma_2 \oplus \cdots \oplus \Gamma_n}$$

(2.) we "solved" the problem using some self-consistent Gaussian / large-N approach

$$\int \mathcal{D}Q \, e^{-\frac{1}{8v}\int \mathrm{tr}\,\hat{Q}\hat{Q} - \frac{1}{2}\int \mathrm{tr}(Qg)} \sim e^{\frac{v}{2}\int \mathrm{tr}(\hat{g}\hat{g})}$$

→ integrate out order parameter

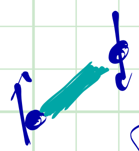
→ saddle point

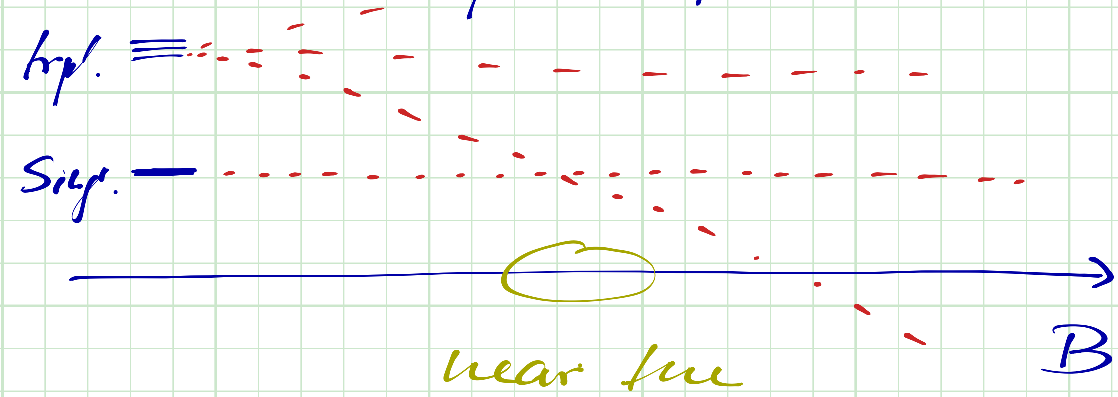


today: we will analyze some problems where (1) is still the main theme, but where we have to do smth. different from (2).

Problem 1: frustrated dimers in a magnetic field

Consider a system of strongly coupled dimers

 $J_0 \vec{S}_1 \cdot \vec{S}_2$



near the crossing of triplets/singlets

near the crossing we have two relevant states

$|\text{singlet}\rangle \longleftrightarrow |0\rangle$ no bosons

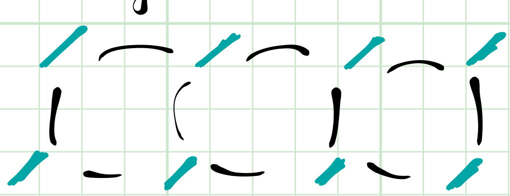
$|m=-1, \text{triplet}\rangle \longleftrightarrow |1\rangle$ one bosons

description in terms of hard-core bosons

$$H = \sum_i (\varepsilon - \mu)\, a_i^\dagger a_i \;+\; U \sum_i n_i (n_i - 1)$$

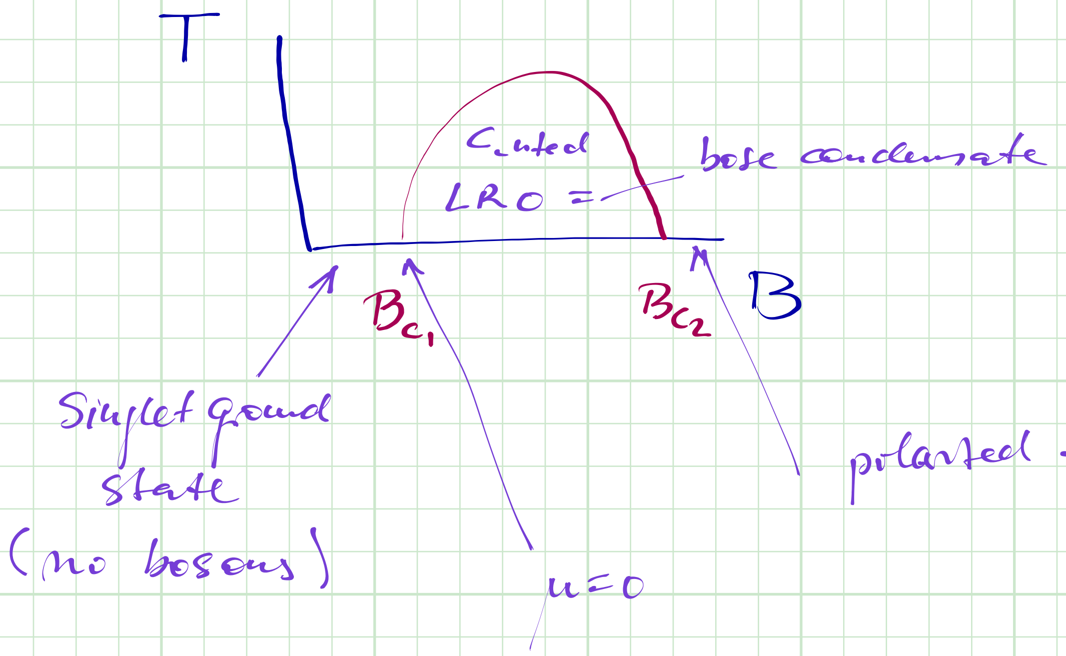
hard core repulsion $U \to \infty$

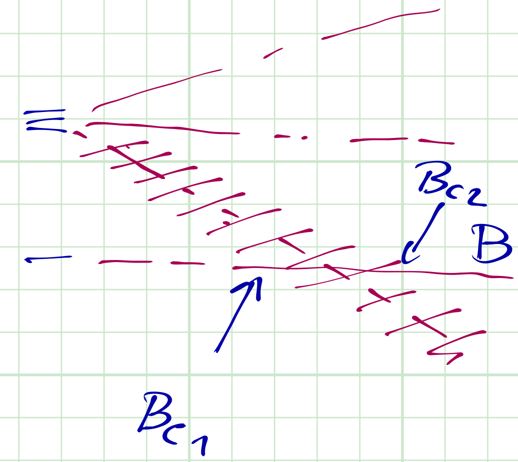
(mapping to pseudospins $\longrightarrow$ hard core bos.)

 inter-dimer exchange $\equiv$ hopping of triplons between sites

$$H = \sum_k (\varepsilon_k - \mu)\, a_k^\dagger a_k \;+\; U \sum_i n_i (n_i - 1)$$

chemical pot.: $\mu = g \mu_B B - J_0 - \min \varepsilon(k)$





This is a rather simple quantum phase transition

$$\varepsilon(k) = \frac{k^2}{2m}$$

$\varepsilon(k) - \mu + u\langle n\rangle$ Hartree approximation

$\Rightarrow$ the condition for BEC is a vanishing chemical potential

$$\mu = u\langle n\rangle = u \int \frac{d^d k}{(2\pi)^d} \frac{1}{e^{\beta_c \varepsilon(k)} -}$$

$\Rightarrow$ $\mu = u\, T_c^{d/2} \cdot \text{const}$ $\qquad k/\sqrt{T_c} = x$

$$\boxed{T_c \sim \mu^{2/d}}$$

$d=2$ const diverges but

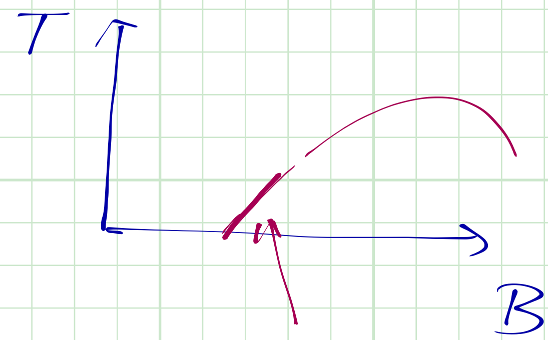
$$T_{BKT} \sim \mu \frac{\log 1/\mu}{\log \log 1/\mu}$$

Popov ('72)
D. Fisher
P. Hohenberg
('88)

$BaCuSi_2O_6$

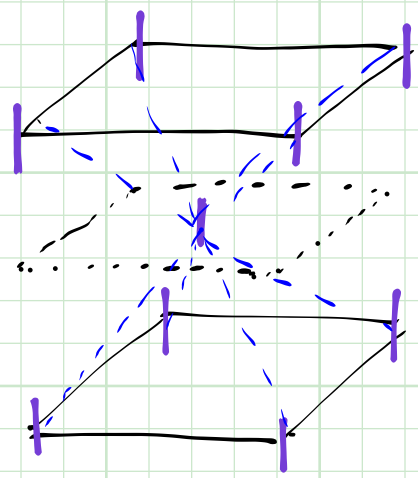
(3-dim system)

S. E. Sebastian
Nature 2006



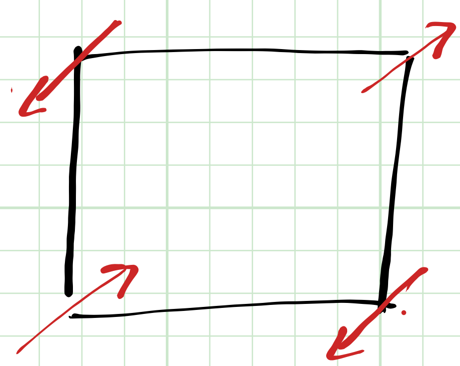
linear (looks like $d=2$)

dimers on a lattice (bct)

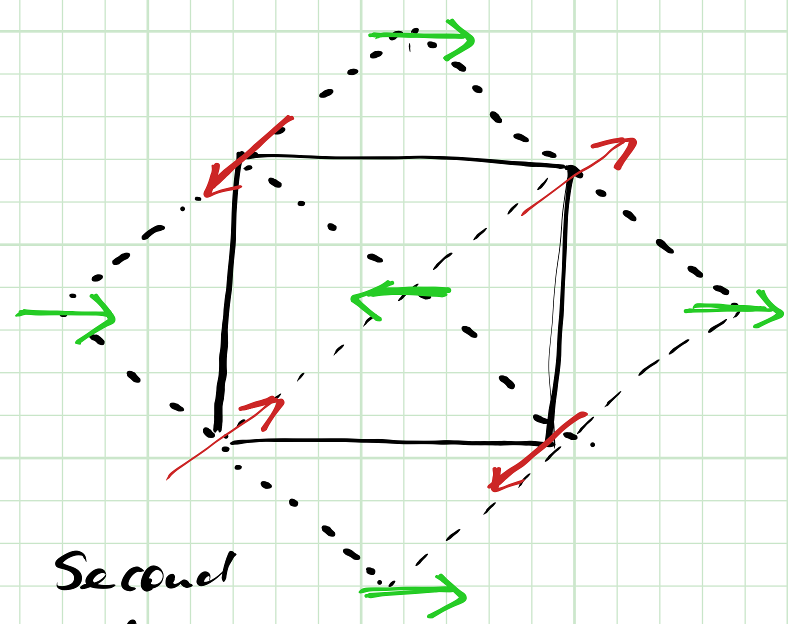


| dimers

: interlayer coupling

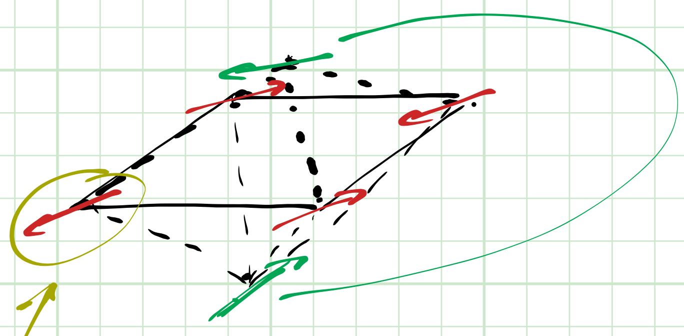


one layer



second layer

- frustrated inter layer coupling
- classical ground state : layers are decoupled.



Coupling only if the "red" spins are not classically ordered

fluctuation induced coupling in the third dimension

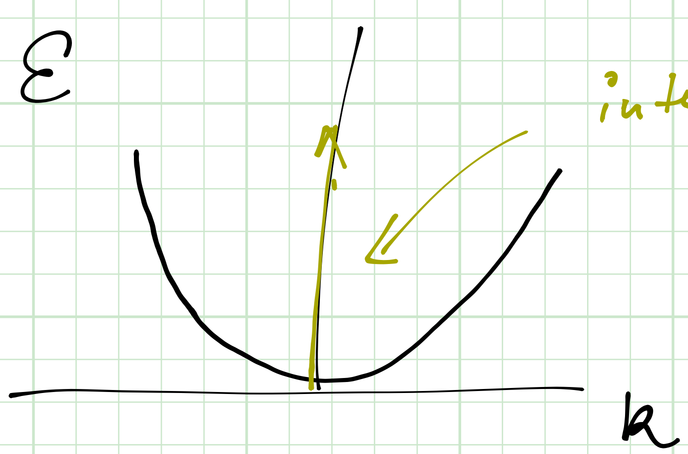
tight binding model of triplets on the bct lattice:

$$E_k = \varepsilon_{k_\parallel} + 2t_\perp \cos k_z \times \underbrace{\cos\frac{k_x}{2}\,\cos\frac{k_y}{2}}_{\gamma(k_\parallel)}$$

$$\varepsilon_{k_\parallel} = t_\parallel(\cos k_x + \cos k_y + 2)$$

ordering vector $\vec{Q} = (\pi, \pi, q_z)$

$$\gamma_{k_\parallel = (\pi,\pi)} = 0$$



interlayer boson hopping vanishes at condensation point let us formulate this problem in a language that is most similar to the nematic order of the iron-based systems

$$a = \frac{1}{\sqrt{2}} \left(\phi_x - i \phi_y \right)$$
$$a^\dagger = \frac{1}{\sqrt{2}} \left(\phi_x + i \phi_y \right)$$

scalar $O(2)$ field

$$S = S_{dyn} + \sum_{\text{layers}} \int_{q_{\parallel}} \left(q_{\parallel}^2 - \mu \right) \left(\vec{\phi}_1 \cdot \vec{\phi}_1 + \vec{\phi}_2 \cdot \vec{\phi}_2 \right)$$

in-plane kinetic energy

$$+ \sum_{\text{layers}} \eta \int_{q_{\parallel}} q_x q_y \, \vec{\phi}_1 \cdot \vec{\phi}_2$$

coupling between layers

$$+ u \sum_{\text{layers}} \int_x \left(\left(\vec{\phi}_1 \cdot \vec{\phi}_1 \right)^2 + \left(\vec{\phi}_2 \cdot \vec{\phi}_2 \right)^2 \right)$$

hard core repulsion

$$+ g \sum_{\text{layers}}' \int_x \left(\vec{\phi}_1 \cdot \vec{\phi}_2 \right)^2$$

inter layer repulsion

$$S_{dyn} = -i \sum_{i=1,2} \int_x \varepsilon_{\alpha\beta} \, \phi_{i\alpha} \, \partial_\tau \, \phi_{i\beta}$$

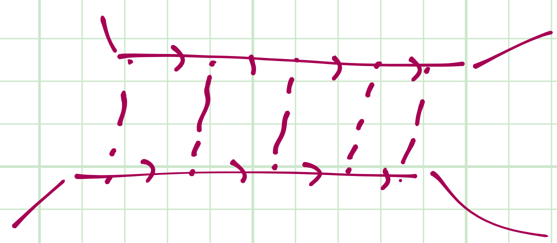
we have bosons!!

$$\left(\neq \sum_{i=1,2} \int_x \left(\partial_\tau \phi_{i\alpha} \right)^2 \right)$$

initial interlayer interaction $g = 0$

$$+ g \sum_{\text{layers}}' \int_x \left(\vec{\phi}_1 \cdot \vec{\phi}_2 \right)^2$$

Solution: low-density expansion

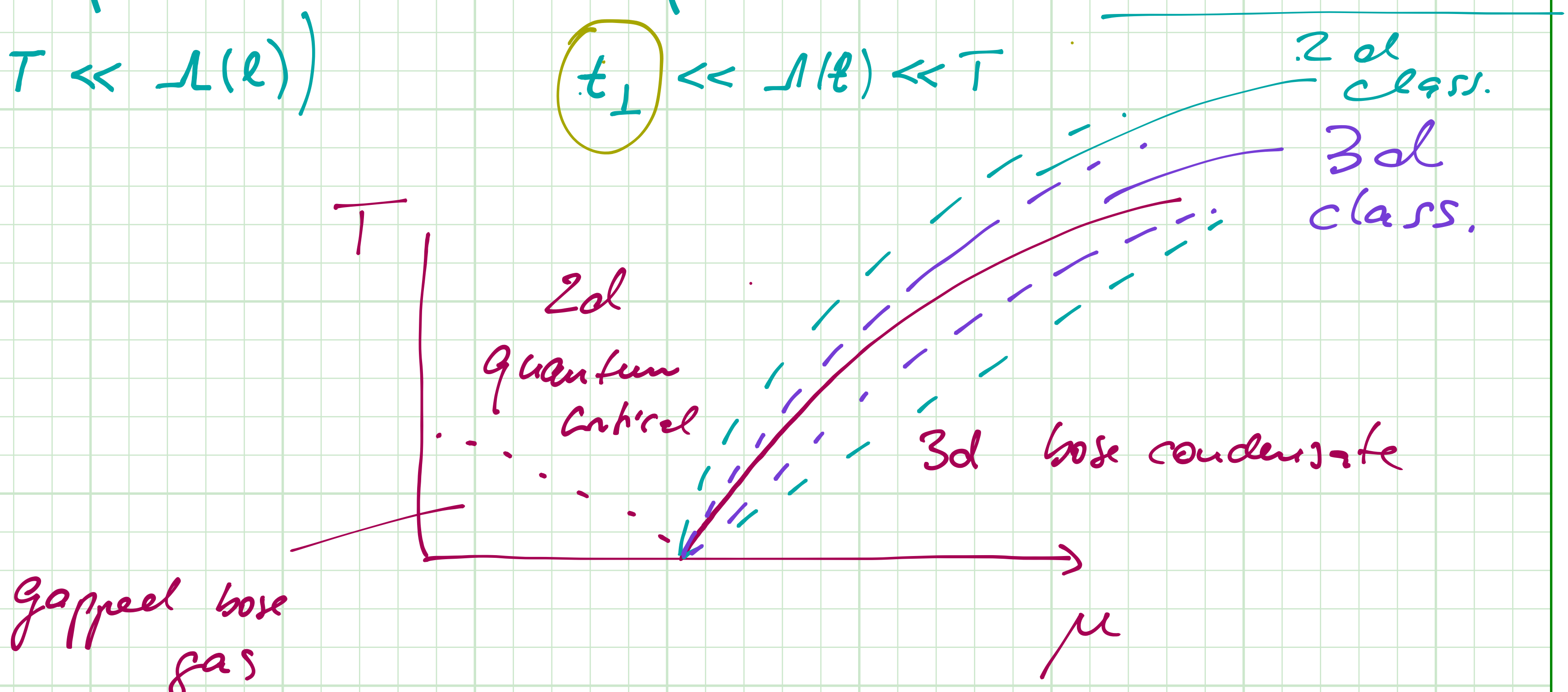


only diagrams with parallel lines!

Can be cast in a renormalization group flow:

pair scatt. between layers generated	single particle coupl. due to thermal excit.	isotropic 3d
2d quantum regime	2d classical regime	3d classical
$(T \ll \Lambda(\ell))$	$t_\perp \ll \Lambda(\ell) \ll T$	

$\longrightarrow \ell$



i) in the quantum regime: effective 2-d behavior

$$\Longrightarrow T_c \sim \mu \, \frac{\log 1/\mu}{\log \log 1/\mu}$$

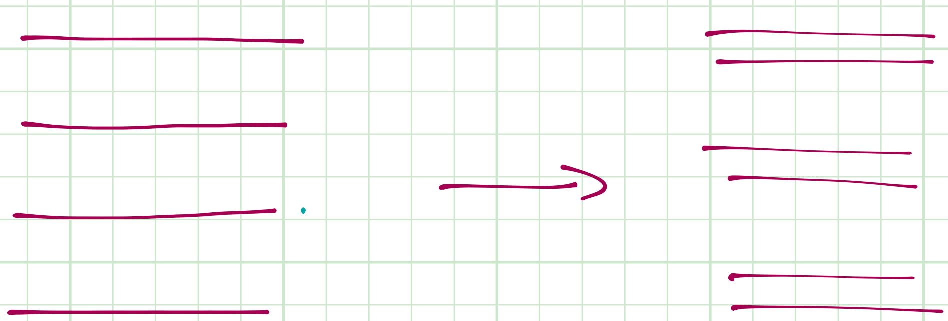
ii) quantum fluctuations generate

$$+ g \sum_{\text{layers}}' \int_x \left(\vec{\phi}_1 \cdot \vec{\phi}_2 \right)^2$$

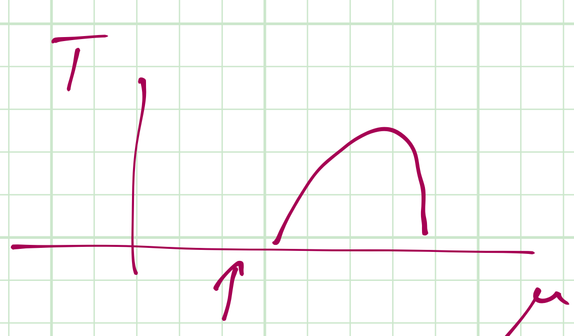
it generates emergent Ising order

$$\varphi = \langle \vec{\phi}_1 \cdot \vec{\phi}_2 \rangle$$

similar to nematic order



dimerization of layers



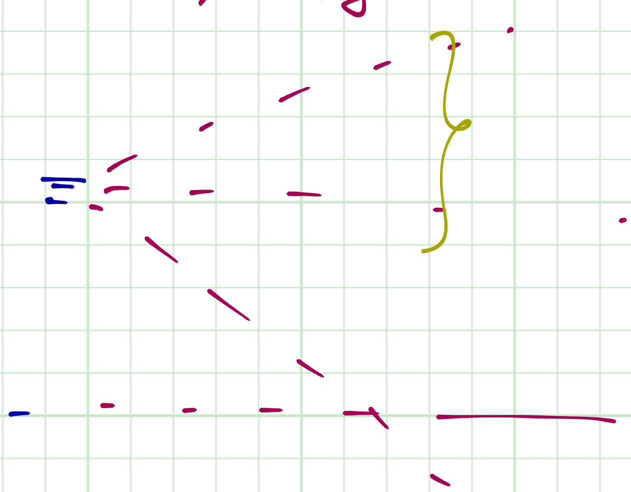
$\Longrightarrow$ fluctuations only at the BEC transition

$\Longrightarrow$ the two transitions are simult.

BEC $\langle a \rangle \neq 0$

dimerz. $\langle \vec{\phi}_1 \cdot \vec{\phi}_2 \rangle \neq 0$

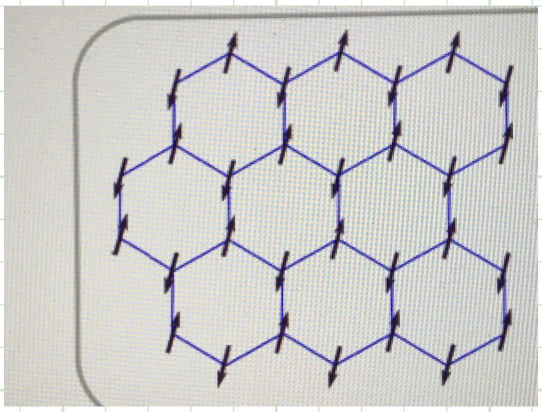
$\Longrightarrow$ complete absence of fluctuations merges the transitions



virtual transitions to other triplet states splits the transition.

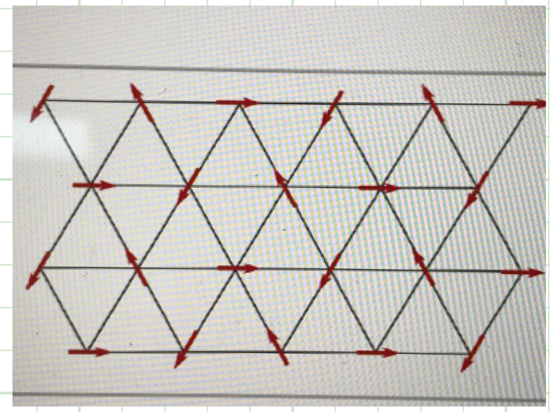
Problem 2 : coupled hexagonal + triangular lattice

hexagonal lattice



co-linear classical ground state

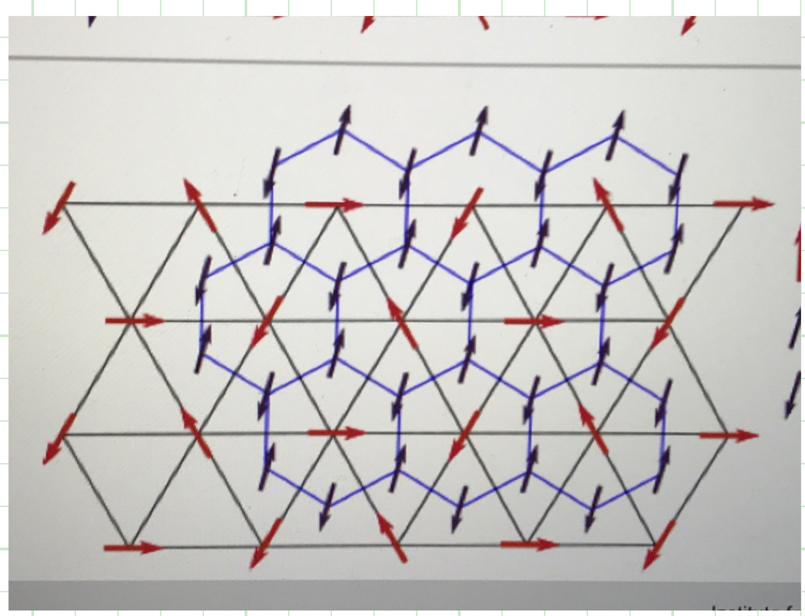
triangular lattice



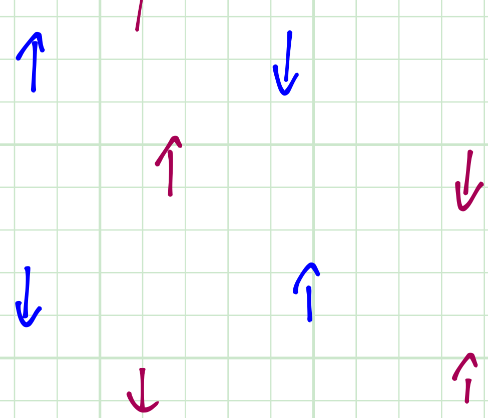
co-planar classical ground state

let us couple these lattices.

mean field of one sub-lattice vanishes in the other sublattice



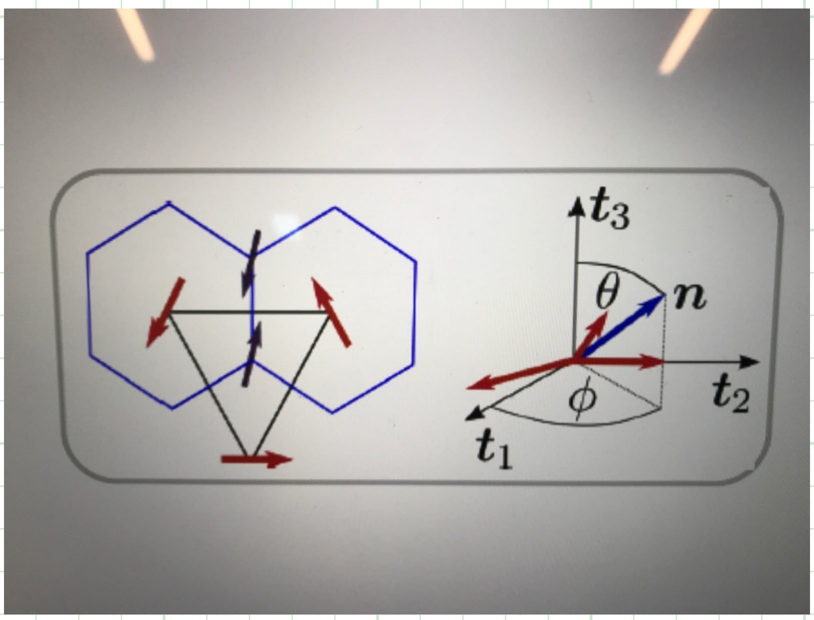
All this is very similar to iron-based systems



$$\vec{m}_1 \cdot \vec{m}_2$$
$$= (\vec{m}_x - \vec{m}_y)\,(\vec{m}_x + \vec{m}_y)$$
$$= (\vec{m}_x^2 - \vec{m}_y^2)$$

emergent order parameter : scalar product of two order-parameters

but: we have co-linear + coplanar order



coplanar order:

$$S_n = \frac{\kappa}{2}\int d^2x \left(\partial_\mu \vec{n}\right)^2$$

($O(3)/O(2)$ nonlin. σ-model)

non-coplanar order

$$S_t = \frac{1}{2}\sum_{i=1}^{3}\int d^2x\, K_i \left(\partial_\mu \vec{t}_i\right)^2$$

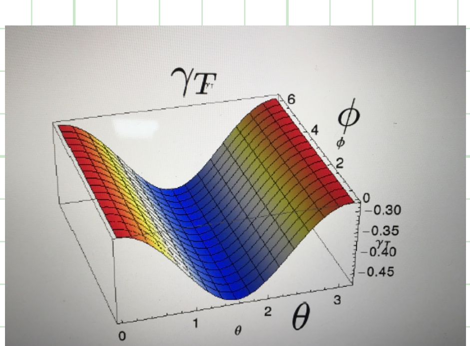
($SO(3)$ nonlin. σ-model)

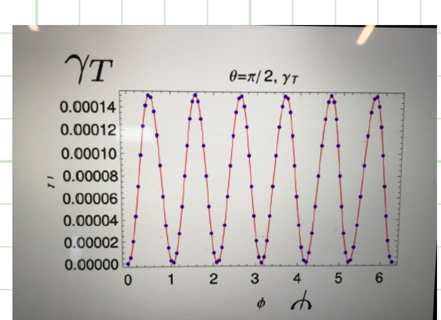
both theories are understood using RG approaches.

question: how are they coupled

answer: via fluctuations (spin waves)

$$\delta E = J\left(\frac{J'}{J}\right)^2 \cos^2\theta + J\left(\frac{J'}{J}\right)^6 \sin^2\theta \sin^3(3\varphi)$$





$$\Rightarrow \quad S_{copl.} = \alpha \int d^2x \, (\vec{n}\cdot\vec{t}_3)^2 + \lambda \int d^2x \left[(\vec{n}\cdot\vec{t}_1)^3 - (\vec{n}\cdot\vec{t}_1)(\vec{n}\cdot\vec{t}_2)^2 \right]$$

$$\uparrow \qquad\qquad\qquad \uparrow$$

$$\cos^2\theta \qquad\qquad\qquad \sin^2(3\varphi)$$

both α, λ are small and strongly relevant

Crossover to coplanar coupling:

$$\frac{d\alpha}{dl} = 2\alpha \qquad \frac{dK}{dl} = -\frac{1}{2\pi}$$

$$\boxed{T_{copl.} \sim \frac{4\pi J}{\log 1/\alpha}}$$

below $T_{copl.}$ $\vec{n}$ is in the plane of the triangular spins

how to describe the coplanar regime?

- the $SO(3)$ spin configuration can be described by three Euler angl.

$SO(3)$ - matrix $\quad t = (\vec{t}_1, \vec{t}_2, \vec{t}_3)$

$$= e^{-i\phi\tau_3}\, e^{-i\theta\tau_1}\, e^{-i\psi\tau_3}$$

- coplanar constraint $\quad t = (\vec{n}, \vec{n}_1, \vec{n}_2)\, e^{i\chi\tau_3}$

$\Rightarrow$ low energy action

$$X = (\underbrace{\phi, \theta, \psi,}_{\text{Euler}} \chi)$$

χ ... relative phase between the sublattices

$$S_{\text{neo}} = \frac{1}{2} \int d^2x \; g_{ij}[X(x)] \, \partial_\mu X^i(x) \, \partial_\mu X^j(x)$$

$$g = \begin{pmatrix} g^{SO(3)} & K^T \\ K & I_\chi \end{pmatrix}$$

Spin-configuration 4-dimensional vec in curved space

= Nambu-Goto action of 4-dimens. target space living in 1+1 dimensions

$\Rightarrow$ Friedan (Ricci flow)

$$\frac{dg_{ij}}{dl} = \frac{1}{2\pi} R_{ij} - \frac{1}{8\pi^2} R_i^{\;klm} R_{jklm}$$

$$(R_{ij} = R^k_{\;ikj} \text{ Ricci tensor})$$

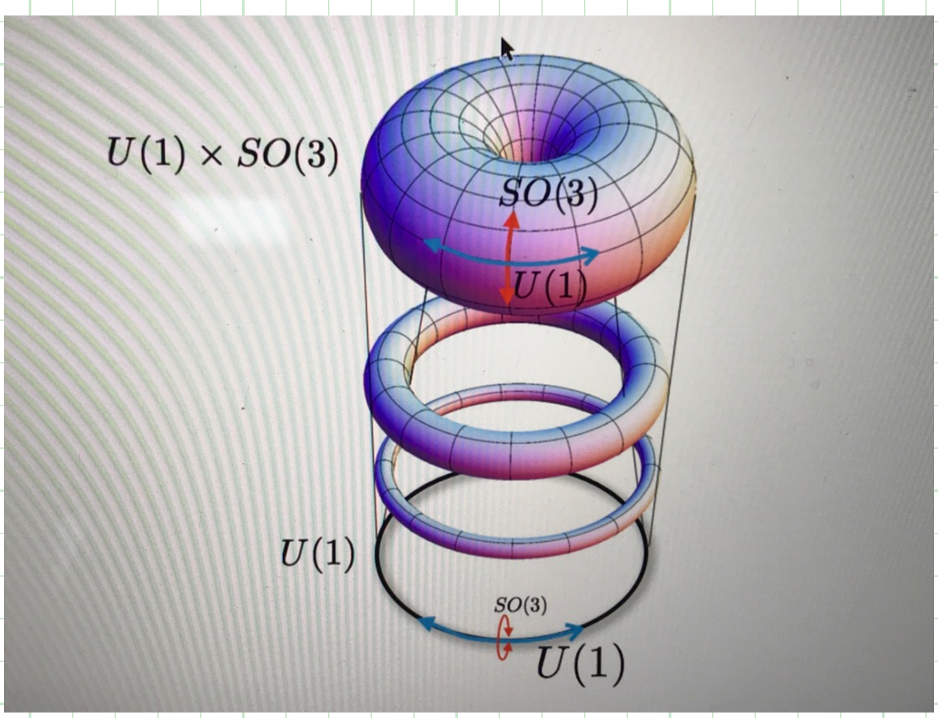
•) the analysis of the flow equations is tedious but straightforward

It follows

stiffness grows

$$g = \begin{pmatrix} g^{SO(3)} & 0 \\ 0 & I_\chi \end{pmatrix}$$

$$\frac{dI_\chi}{dl} = 0$$



the key cond. is that the relative angle decouples

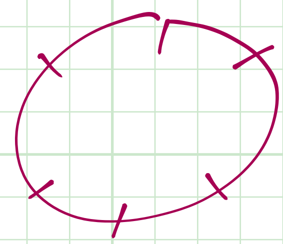
$\Longrightarrow$ effective low-energy theory

$$S_\chi = \frac{I_\chi^*}{2}\int d^2x \, (\partial_\mu \chi)^2 + \frac{\lambda^*}{2}\int d^2x \cos(6\chi)$$

$I_\chi^* \simeq \frac{1}{8} I_\chi \qquad \lambda^* \sim (\partial'/\partial)^2$

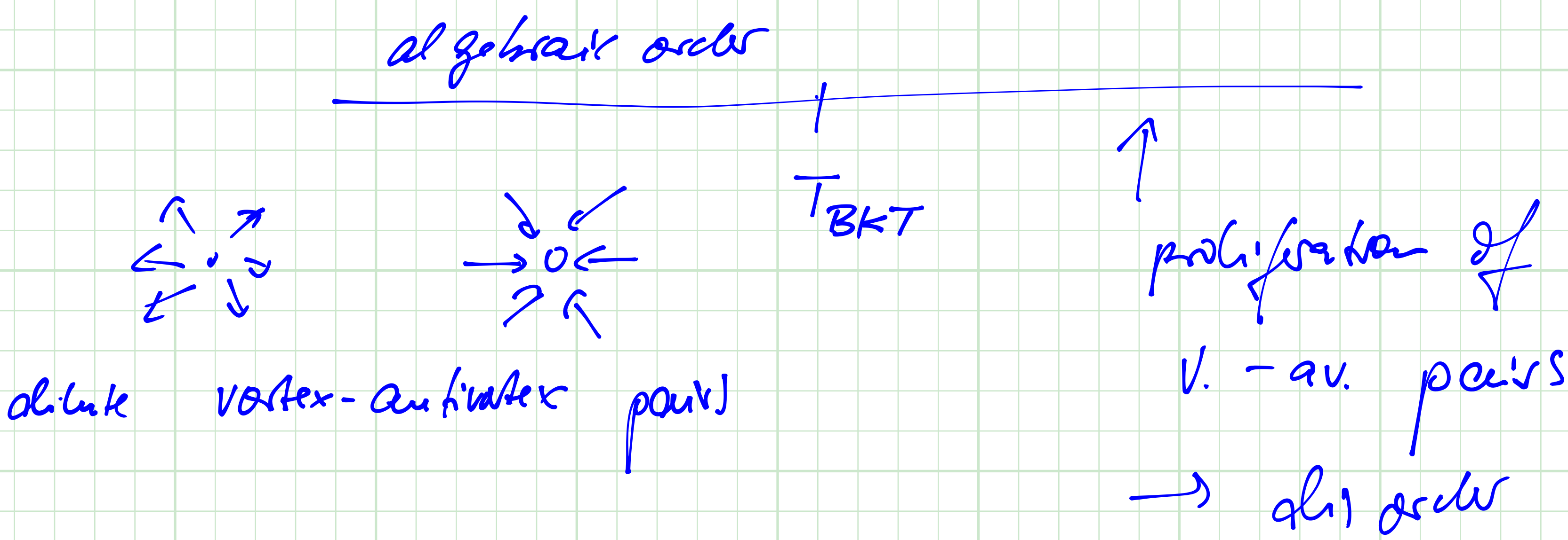
$$S_\chi = \frac{T_\chi^*}{2}\int d^2x\,(\partial_\mu \chi)^2 + \frac{\lambda^*}{2}\int d^2x\,\cos(6\chi)$$

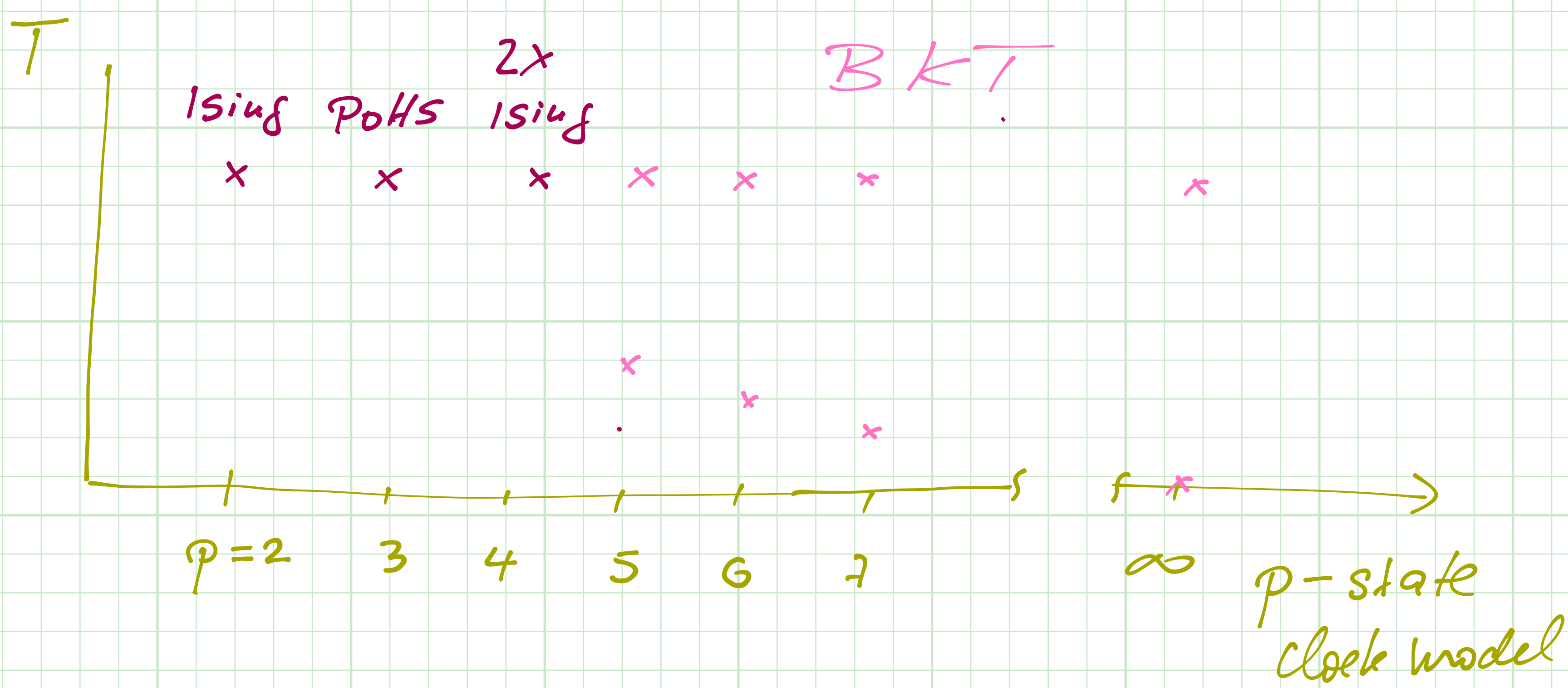
this is the six-state clock model



planar vector with six positions

·) $\lambda^* = 0$ this is the xy model





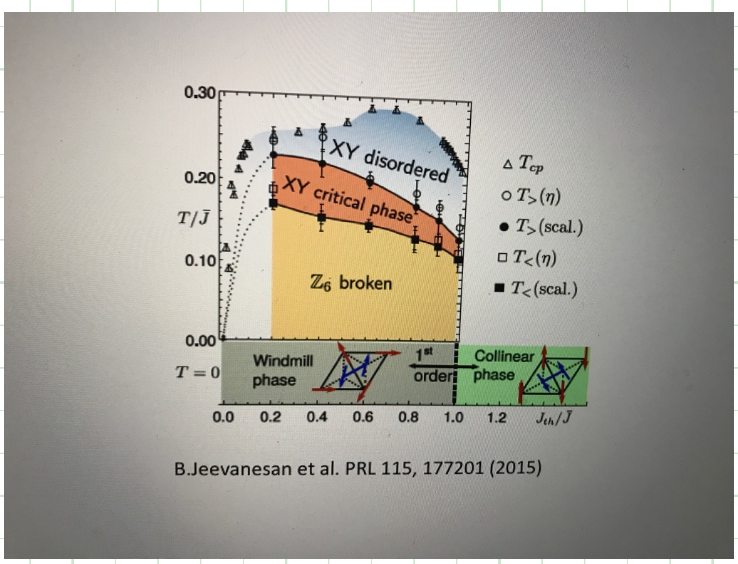
broken Z_6 | algebr. order

$T^{<}_{BKT}$ | $T^{>}_{BKT}$ | T

two BKT - transitions in a two-dimensional Heisenberg model

Order parameter $\qquad \vec{n} \cdot \vec{t}_2 = \cos(\chi)$

relative orientation of two sublattices



Monte Carlo simulation