Summary of yesterday's lecture

Composite order parameter

$$\varphi = \sum_{\mu\nu} \tau_{\mu\nu} \, \eta_\mu \eta_\nu$$

$$\underset{\substack{\parallel \\ A_1}}{\Gamma \otimes \Gamma} = \Gamma_1 \oplus \underbrace{\Gamma_2 \oplus \dots \Gamma_n}_{\text{non trivial bilinear forms}}$$

$\Rightarrow$ $\langle \varphi \rangle \neq 0$ breaks a new symmetry

So far we focussed on magnetism.

Let us analyze whether similar behavior occurs in superconductors.

the superconducting order parameter
(another detour)

pairing part of the Hamiltonian:

$$H_\Delta = \sum_k \left(\psi^\dagger_{ka} \, \underset{\substack{\uparrow \\ \text{spin, orbitals, subbands} \dots}}{\Delta_{ab}(k)} \, \psi^\dagger_{-kb} + h.c. \right)$$

$$\Delta_{ab}(k) \sim \langle \psi_{ka} \psi_{-kb} \rangle \underset{\substack{\uparrow \\ \text{Pauli} \\ \text{principle}}}{=} - \langle \psi_{-kb} \psi_{ka} \rangle$$

$$= - \Delta_{ba}(-k)$$

expand in spin space $a = (\alpha, s)$

$$\Delta_{\alpha s, \beta s'}(k) = \Delta^0_{\alpha\beta}(k) \, \underset{\substack{\uparrow \\ \text{singlet}}}{i \sigma^y_{ss'}} + \underset{\substack{\uparrow \\ \text{triplet}}}{i \vec{d}_{\alpha\beta} \cdot \vec{\sigma}_{ss''} \, \sigma^y_{s''s'}}$$

let us ignore the additional d.o.f (α, β)

Space inversion: if only $\vec{k} \to -\vec{k}$

$$\Delta^0(k) = \Delta^0(-k) \qquad \vec{d}(k) = -\vec{d}(-\vec{k})$$

(antisymmetry from spin) (a.s. from space)

• parity:

$$P\,\Delta_{ss'}(k) = \Delta^0(-k)\, i\sigma^y_{ss'} + i\vec{d}(-\vec{k})\cdot(\vec{\sigma}\,\sigma^y)_{ss'}$$
$$= \Delta^0(k)\, i\sigma^y_{ss'} - i\vec{d}(k)\,(\vec{\sigma}\,\sigma^y)_{ss'}$$

if the system is inversion symmetric

$$P\,\Delta_{ss'}(k) = \pm\,\Delta_{ss'}(k)$$
(well defined parity)

System with inversion symmetry is either singlet or triplet!

other symmetries spinor $\psi \to R_\psi(g)\,\psi$

spinor representation group element

$\hat{\Delta}(k)$ transforms as $\psi_k\,\psi^T_{-k}$

$$\Delta(k) \xrightarrow{g} R_\psi(g)\,\Delta\left(R_0^{-1}(g)\vec{k}\right) R_\psi^T(g)$$

coordinate representation

expand $\Delta(k)$ in terms of basis functions χ_μ^n transforming under the irreducible representation of the group.

$$\Delta(k) = \sum_n \sum_{\mu=1}^{d_n} \eta_\mu^n \, \chi_\mu^n(k)$$

$\in \mathbb{C}$

(the order parameter)

matrices in spin, orbital...

$$R_\psi(g)\, \chi_\mu^n(R_v^{-1} k)\, R_\psi^T(g) = \left(R_\chi^n(g)\right)_{\mu\nu} \chi_\nu^n(k)$$

mixes only within the irreduc. rep.

bottom line

$$\eta_\mu^n \xrightarrow{g} \eta_\nu^n \, R_\chi^n(g)_{\nu\mu}$$

invariance of the free energy:

$$\mathcal{F}(\eta_\mu^n, (\eta_\mu^n)^*) = \mathcal{F}(\eta_\mu^n e^{i\varphi}, (\eta_\mu^n)^* e^{-i\varphi}) \quad \forall \varphi \in \mathbb{R}$$

$$\mathcal{F}(\eta_\mu^n, (\eta_\mu^n)^*) = \mathcal{F}(\eta_\nu^n R_\chi^n(g)_{\nu\mu}, \eta_\nu^n R_\chi^n(g)_{\nu\mu}^*) \quad \forall g \in$$

let us expand to second order

$$\mathcal{F}\left(\eta_\mu^n,(\eta_\mu^n)^*\right) = \mathcal{F}(0,0)$$

$$+\sum_{n,n'}\sum_{\mu=1}^{d_n}\sum_{\mu'=1}^{d_{n'}} (\eta_\mu^n)^* \mathcal{M}_{\mu\mu'}^{nn'} \eta_{\mu'}^{n'} + \dots$$

to satisfy the symmetry condition of the free energy, it must hold

$$\mathcal{M}^{nn'} = R_\chi^n(g)^* \mathcal{M}^{nn'} \left(R_\chi^{n'}(g)\right)^T$$

this must be true for all group elements g

$\Rightarrow$ we can sum over all g and use the "Grand orthogonality theorem"

$$\sum_{g\in G} R^n(g)^*_{\mu\nu} R^{n'}(g)_{\mu'\nu'} = \frac{|G|}{d_n}\delta_{nn'}\delta_{\mu\mu'}\delta_{\nu\nu'}$$

$$\Rightarrow \quad \mathcal{M}_{\mu\mu'}^{nn'} = a_n\,\delta_{\mu\mu'}\delta_{nn'}$$

$$\mathcal{F}\left(\eta_\mu^n,(\eta_\mu^n)^*\right) = \mathcal{F}(0,0) + \sum_n \sum_{\mu=1}^{d_n} a_n(T)\,|\eta_\mu^n|^2 + \dots$$

1.) the expansion coefficients for all components of η_μ^n within the same irreducible representation are the same!

2.) the n for which $a_n(T)<0$ first is the one that orders

$\Longrightarrow$ Symmetry breaking takes place according to the irreduc. representations of the symmetry group of the system.

Now we know a little bit about a symmetry motivated description of superconductors.

let us now look at superconductors from the perspective of yesterday's talk.

lets look at the group $\underline{D_{4h} \otimes U(1)}$

there are two 2-dimensional irreducible representations E_g, $\underline{\underline{E_u}}$

this is odd under parity i.e. what we need for triplets

$$\Delta_{ss'}(k) = \left(\vec{d}_k \, \vec{\sigma} \, i\sigma_y\right)_{ss'}$$

in the supercond. Sr_2RuO_4 a frequently discussed pairing state is the $S=0$ triplet

$$\vec{d}_k = \vec{e}_z \left(\Delta_x \sin k_x + \Delta_y \sin k_y\right)$$

(frequently $(\Delta_x k_x + \Delta_y k_y)$)

(Δ_x, Δ_y) are complex degrees of freedom that transform according to E_u

Which bilinear forms can we generate?

$$E_u \otimes E_u = A_{1g} \oplus \underbrace{A_{2g} \oplus B_{1g} \oplus B_{2g}}_{\text{non-trivial bilinear forms}}$$

Let us analyze the allowed bilinear forms

$$\varphi = \sum_{\mu\nu}{}' \Delta_\mu^* \tau_{\mu\nu} \Delta_\nu$$

unit matrix + Pauli matrices because of two-dim repr.

(total 4 matrices)

Suppose we have a d_n-dimensional irreducible representation $\Rightarrow$ d_n^2 indep. matrices

$d_n = 3$: 8 Gell-Mann matrices + unit matrix explicitly

$$\varphi_0 = \Delta^\dagger \tau^0 \Delta = \Delta_x^* \Delta_x + \Delta_y^* \Delta_y \qquad A_{1g}$$

$$\varphi_1 = \Delta^\dagger \tau^x \Delta = \Delta_x^* \Delta_y + \Delta_y^* \Delta_1 \qquad B_{2g}$$

$$\varphi_2 = \Delta^\dagger \tau^y \Delta = i\left(\Delta_y^* \Delta_x - \Delta_x^* \Delta_y\right) \qquad A_{2g}$$

$$\varphi_3 = \Delta^\dagger \tau^z \Delta = \Delta_x^* \Delta_x - \Delta_y^* \Delta_y \qquad B_{1g}$$

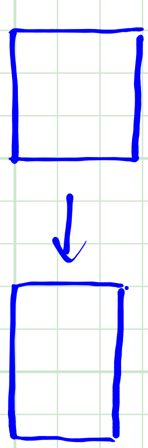
since $\Delta_x \sim x \quad (k_x)$

$\Delta_y \sim y \quad (k_y)$

B_{1g}

$\Delta_x^* \Delta_x - \Delta_y^* \Delta_y$

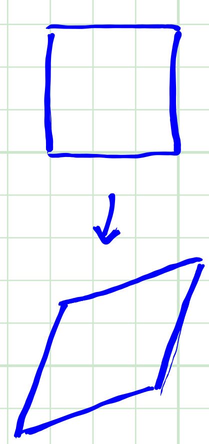
$x^2 - y^2$ nematic



B_{2g}

$\Delta_x^* \Delta_y + \Delta_y^* \Delta_x$

xy nematic



rotational symmetries broken

A_{2g}

$i(\Delta_x^* \Delta_y - \Delta_y^* \Delta_x)$

$-i(\Delta_y^* \Delta_x - \Delta_x^* \Delta_y)$

$\langle \Delta_x^* \Delta_y \rangle \neq \langle \Delta_x^* \Delta_y \rangle^*$

$\downarrow$

time reversal symmetry breaking

to actually "solve" the problem, we need to understand the quartic terms of the G.-L. expansion

$$\varphi^{n,k} = \sum_{\alpha\beta} \Delta^*_\alpha \, \tau^{n,k}_{\alpha\beta} \, \Delta_\beta$$

n: irred. rep.
$k = (1,\dots d_n)$

groups of the $\varphi^{n,k}$ transform according to irreducible representations

$$\Gamma \otimes \Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \dots \Gamma_m$$

Γ_n :

$$R^\Gamma(g) \, (\tau^{n,1}, \tau^{n,2}, \dots)_k \, R^\Gamma(g)^{-1}$$

$$= \underline{\underline{(R^n_\tau)^{-1}_{kp}}} \, (\tau^{n,2}, \tau^{n,2}, \dots)_p$$

transformation of bilinears

$$F^{(4)} = \sum_{lmrs} u_{lmrs} \, \Delta^*_l \Delta_m \Delta^*_r \Delta_s$$

expand w.r.t. the first and second index pair

$$u_{lmrs} = \sum_{ij} u_{ij} \, \tau^i_{lm} \, \tau^j_{rs}$$

$$\Rightarrow \quad F^{(4)} = \sum_{n,n'}' \sum_{\mu,\mu'}' u^{kk'}_{nn'} \underbrace{\left(\Delta^{+} \tau^{n,k} \Delta\right)}_{\varphi^{n,k}} \underbrace{\Delta^{+} \tau^{n'k'} \Delta}_{\varphi^{n'k'}}$$

we know how this transforms

$\Rightarrow$ the symmetry of the free energy yields

$$u^{kk'}_{nn'} = \sum_{pp'}' R^{n}(g)^{-1}_{pk} \, u^{pp'}_{nn'} \, R^{n'}(g)^{-1}_{p'k'}$$

we sum over all group elements $\in G$

$$\Rightarrow u^{kk'}_{nn'} = \frac{1}{d_n} \delta_{nn'} \delta_{kk'} \sum_{p}' u^{pp'}_{nn'}$$

$$\Rightarrow \boxed{u^{kk'}_{nn'} = u_n \, \delta_{nn'} \, \delta_{kk'}}$$

independent on k

$$\boxed{F^{(4)} = \sum_{n}' u_n \sum_{k}^{+} \varphi^{n,k} \varphi^{n,k}}$$

quartic interactions are the same for all bilinear forms of the same irr. rep.

not all terms are independent (Fierz identities)

fundamental repr. of $SU(N)$ τ^i

$$\mathrm{tr}(\tau^i \tau^j) = 2\delta^{ij}$$

$$\Rightarrow \frac{1}{2}\sum_{i=1}^{N^2-1} \tau^i_{\alpha\beta}\,\tau^i_{\gamma\delta} = \delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{1}{N}\delta_{\alpha\beta}\delta_{\gamma\delta}$$

$$\Longrightarrow \sum_{i=1}^{N^2-1} (\Delta^+ \tau^i \Delta)(\Delta^+ \tau^i \Delta)$$

$$= \sum_{\substack{\alpha\beta\\ \gamma\delta}} \sum_{i=1}^{N^2-1} \Delta^*_\alpha \tau^i_{\alpha\beta} \Delta_\beta \, \Delta^*_\gamma \tau^i \Delta_\delta$$

$$= \sum_{\alpha\beta\gamma\delta} \Delta^*_\alpha \Delta_\beta \, \Delta^*_\gamma \Delta_\delta \left(2\delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{2}{N}\delta_{\alpha\beta}\delta_{\gamma\delta}\right)$$

$$= \sum_{\alpha\beta} \Delta^*_\alpha \Delta_\alpha \, \Delta^*_\beta \Delta_\beta \times \left(2 - \frac{2}{N}\right)$$

$$= \left(2 - \frac{2}{N}\right)(\Delta^+ \tau^0 \Delta)(\Delta^+ \tau^0 \Delta)$$

back to our problem:

$$F^{(4)} = u_0 \varphi_0^2 + u_1 \varphi_1^2 + u_2 \varphi_2^2$$

$$= (u_0 + u_1)\varphi_0^2 - u_1 \varphi_3^2 + (u_1 + u_2)\varphi_2^2$$

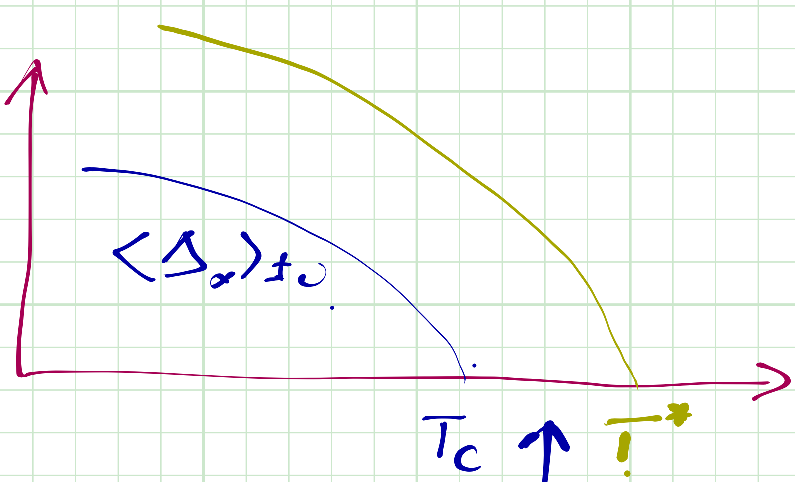
if any of the $u_{i\neq 0} < 0$ we have a chance to get an preemptive transition (we can condense the H.S. field)

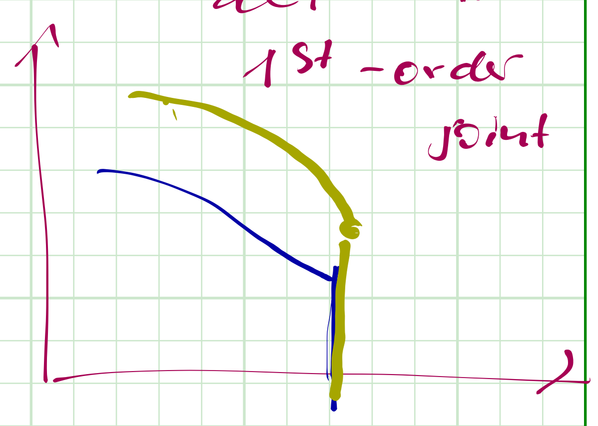
Suppose $u_2 > 0$ (no TRS-breaking)

if $u_1 < 0 \Rightarrow \varphi_1$ can condense
(xy-nematic)

if $u_1 > 0 \Rightarrow \varphi_3$ can condense
(x^2-y^2 nematic)

whatever happens, we find





$\Rightarrow$ there is no direct second order transition to a s.c. with $d_n > 1$!!

Caveat: - this is a fluctuation effect!

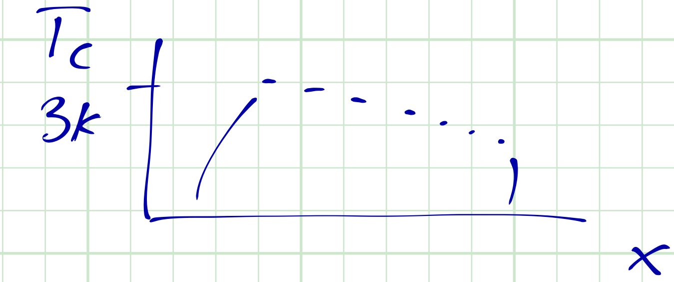
- usually superconductors with $k_F \xi \gg 1$ don't fluctuate!

→ there are many new low-density superconductors with unconventional pairing

Example: $Cu_x Bi_2 Se_3$

(doped topological insulator)

$Cu_x Bi_2 Se_3$ point group D_{3d}



Fu + Berg : E_u representation

$$E_u \otimes E_u = A_{1g} \oplus A_{2g} \oplus E_g$$

TRS-break. nematic (itself 2-dim)

$$F^{(4)} = u\left(\Delta^\dagger \Delta\right)^2 + v\left(\Delta^\dagger \tau^y \Delta\right)^2$$

$$= (u+v)\left(\Delta^\dagger \Delta\right)^2 - v\left[\left(\Delta^\dagger \tau^x \Delta\right)^2 + \left(\Delta^\dagger \tau^z \Delta\right)^2\right]$$

nematic tensor

$$q_{\alpha\beta} = \left(\Delta^\dagger \tau^x \Delta\right) \tau^x_{\alpha\beta} + \left(\Delta^\dagger \tau^z \Delta\right) \tau^z_{\alpha\beta}$$

at mean field $(\Delta_x, \Delta_y) = \Delta(\cos\theta, \sin\theta)$

$$\Longrightarrow \quad \vec{n} = (\cos\theta, \sin\theta) \qquad \text{director}$$

$$\langle q_{\alpha\beta}\rangle = q\left(n_\alpha n_\beta - \frac{1}{2}\delta_{\alpha\beta}\right)$$

true nematic

$$S^{(4)} = \frac{u+v}{2}\int(\Delta^\dagger\Delta)^2 - \frac{v}{2}\int \mathrm{tr}(\hat{q}\hat{q})$$

H. S. transformation

$$\int \mathcal{D}Q \; e^{-\frac{1}{8v}\int \mathrm{tr}(\hat{Q}\hat{Q}) - \frac{1}{2}\int \mathrm{tr}(\hat{Q}\hat{q})}$$

$$\sim e^{\frac{v}{2}\int \mathrm{tr}(\hat{q}\hat{q})}$$

collective variable is a quadrupolar tensor (it transforms under E_g)

$$\Longrightarrow \quad S = \frac{1}{8v}\int_x \mathrm{tr}(\hat{Q}\hat{Q}) - \frac{1}{8u'}\int_x \lambda^2$$

$$+ \int_p \Delta^\dagger \chi(\vec{p}) \Delta$$

$$\chi^{-1} = (r_0 + \underline{\underline{\lambda}} + f(\vec{p}))\tau_0 + \underline{\underline{\hat{Q}}} + \vec{f}\cdot\vec{\tau}$$

we integrate out the order-parameter field and obtain

$$S = \frac{1}{8v}\int_x \mathrm{tr}\,\hat{Q}\hat{Q} - \int_x \frac{1}{u'}\lambda^2 + \int_p \mathrm{tr}\log \chi^{-1}(p)$$

now we take the saddle point approximation (self-consistent Gaussian fluct.)

$$r = r_0 + 2u'\int_p \mathrm{tr}\,(\chi(p))$$

$$Q_{\alpha\beta} = -2v\int_p \mathrm{tr}\left(\chi(p)\cdot\vec{\tau}\right)\cdot\vec{\tau}_{\alpha\beta}$$